1. Both parts of this problem are about iterated double integrals.

(a) (5 points) Evaluate the double integral

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x)\,\cos(y)\,dx\,dy.$$

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x)\cos(y)\,dx\,dy = \int_0^{\pi/2} -\cos(x)\cos(y)\Big|_{x=0}^{x=\pi/2} dy$$

$$= \int_0^{\pi/2} \cos(y)\,dy = \sin(y)\Big|_{y=0}^{y=\pi/2} = \boxed{1}$$

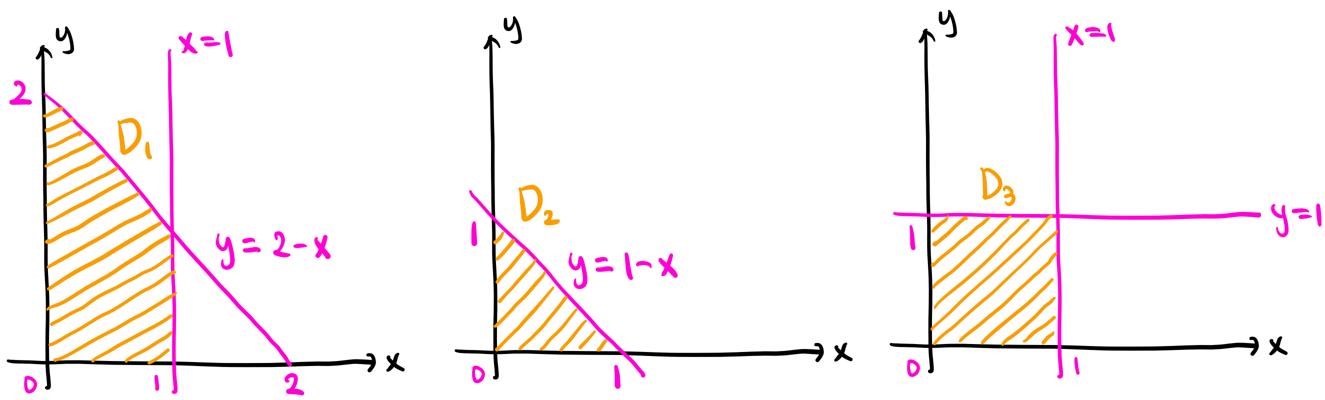
(b) (5 points) Arrange the following three double integrals in order from least to greatest:

$$\int_0^1\int_0^{2-x} e^{x^2+y^2}\,\cancel{dx\,dy}\ dy\,dx,\quad \int_0^1\int_0^{1-x} e^{x^2+y^2}\,\cancel{dx\,dy}\ dy\,dx,\quad \int_0^1\int_0^1 e^{x^2+y^2}\,\cancel{dx\,dy}\ dy\,dx.$$

Explain briefly.

*This problem contains typos as indicated above.

We can sketch the domains of the three integrals as follows:



Since the integrand $e^{x^2+y^2}$ is always positive, its integral gets bigger as the domain gets bigger

$$\Rightarrow \boxed{\int_0^1\int_0^{1-x} e^{x^2+y^2}\,dy\,dx \le \int_0^1\int_0^1 e^{x^2+y^2}\,dy\,dx \le \int_0^1\int_0^{2-x} e^{x^2+y^2}\,dy\,dx}$$

2. Let $f(x, y) = x^2 - xy + y^2 - 3x$.

(a) (5 points) Find all the critical points of f in the x-y plane.

$\nabla f = (f_x, f_y) = (2x - y - 3, -x + 2y)$

$\nabla f = \vec{0} \Rightarrow 2x - y - 3 = 0$ and $-x + 2y = 0 \Rightarrow x = 2, y = 1$

$\Rightarrow$ The only critical point of f is at $\boxed{(2,1)}$

(b) (5 points) Determine whether each critical point is a minimum, maximum, or saddle.

The Hessian of $f(x,y)$ is

$$H = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = f_{xx} \cdot f_{yy} - f_{xy}^2.$$

$$f_{xx} = \frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x}(2x - y - 3) = 2.$$

$$f_{xy} = \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}(2x - y - 3) = -1.$$

$$f_{yy} = \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y}(-x + 2y) = 2$$

$\Rightarrow H = 2 \cdot 2 - 1^2 = 3 > 0, \quad f_{xx} = 2 > 0$

$\Rightarrow$ The critical point at $(2,1)$ is $\boxed{\text{a local minimum.}}$

<u>Note</u> You can also write

$$f(x,y) = \frac{1}{4}x^2 - xy + y^2 + \frac{3}{4}x^2 - 3x = \left(\frac{x}{2} - y\right)^2 + 3\left(\frac{x}{2} - 1\right)^2 + 3$$

and find that $f(x,y)$ attains the global minimum of 3

at $(2,1)$ with $\frac{x}{2} - y = 0$ and $\frac{x}{2} - 1 = 0$.

3. Consider the plane $2x - y + 2z = 9$ and the sphere $x^2 + y^2 + z^2 = 25$.

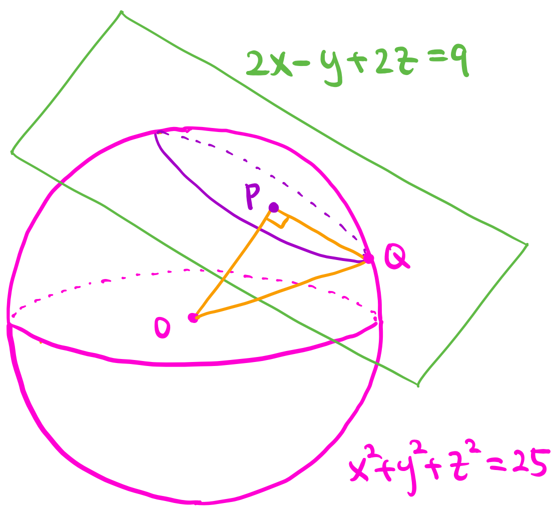
(a) (2 points) Find the distance of the origin $(0, 0, 0)$ from the plane. Conclude that the plane intersects the sphere.

The distance from $(0,0,0)$ to the plane $2x-y+2z-9=0$ is

$$\frac{|2\cdot 0 - 0 + 2\cdot 0 - 9|}{\sqrt{2^2+(-1)^2+2^2}} = \frac{9}{3} = \boxed{3}$$

The plane intersects the sphere $x^2+y^2+z^2=25$ since its distance from the center $(0,0,0)$ is less than the radius 5.

(b) (4 points) The plane intersects the sphere in a circle. Find the area of this circle of intersection.



O : the origin

P : the center of the intersection circle

Q : a point on the intersection circle

$\Rightarrow |\overrightarrow{OP}| = 3$, $|\overrightarrow{OQ}| = 5$ (radius)

(a)

$\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are perpendicular

$\Rightarrow$ The radius of the intersection circle is

$$|\overrightarrow{PQ}| = \sqrt{|\overrightarrow{OQ}|^2 - |\overrightarrow{OP}|^2} = \sqrt{5^2-3^2} = 4$$

$\Rightarrow$ The area of the intersection circle is $\pi \cdot 4^2 = \boxed{16\pi}$

(c) (4 points) Find the center of the circle of intersection.

$\overrightarrow{OP}$ is a normal vector of the plane $2x-y+2z=9$.

$\Rightarrow \overrightarrow{OP} = t(2,-1,2) = (2t, -t, 2t)$ for some t.

P is on the plane $2x-y+2z=9$.

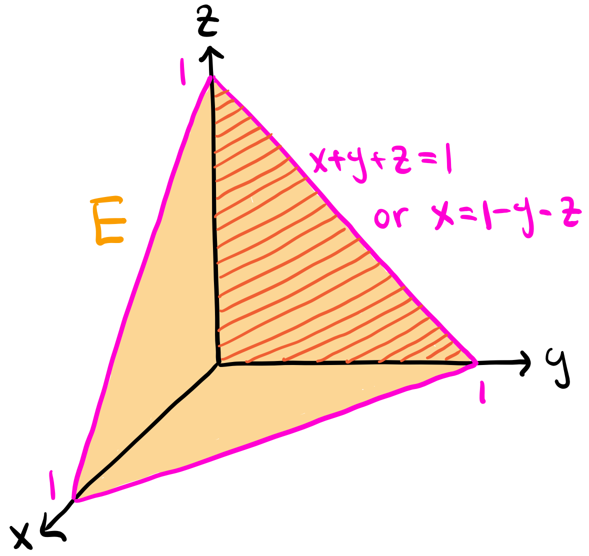
$\Rightarrow 2\cdot 2t - (-t) + 2\cdot 2t = 9 \Rightarrow 9t = 9 \Rightarrow t = 1$.

$\Rightarrow P = \boxed{(2,-1,2)}$

4. Consider the solid tetrahedron with vertices

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1).$$

(a) (6 points) Suppose the density is $\rho(x,y,z) = 1 - z$ (mass per unit volume). Find the total mass of the tetrahedron.



The tetrahedron E is bounded by the planes $x=0,\ y=0,\ z=0,\ x+y+z=1$.

* As a general tip, the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ has x-intercept a, y-intercept b, z-intercept c.

The shadow on the yz-plane: $0 \le z \le 1,\ 0 \le y \le 1-z$

For each point on the shadow: $0 \le x \le 1-y-z$

$\Rightarrow$ E is given by $0 \le z \le 1,\ 0 \le y \le 1-z,\ 0 \le x \le 1-y-z$

$$m = \iiint_E \rho(x,y,z)\,dV = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} 1-z \; dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-z} (1-y-z)(1-z)\,dy\,dz = \int_0^1 \left((1-z)y - \frac{y^2}{2}\right)(1-z)\Big|_{y=0}^{y=1-z} dz$$

$$= \int_0^1 \frac{(1-z)^3}{2}\,dz \underset{\substack{\uparrow \\ u=1-z}}{=} \int_1^0 \frac{u^3}{2}\cdot(-1)\,du = -\frac{u^4}{8}\Big|_{u=1}^{u=0} = \boxed{\frac{1}{8}}$$

(b) (4 points) Assuming the same density as in part (b), find the z-coordinate of the center of mass.

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV = 8\int_0^1 \int_0^{1-z} \int_0^{1-y-z} z(1-z)\,dx\,dy\,dz$$

$$= 8\int_0^1 \int_0^{1-z} (1-y-z)(1-z)z\,dy\,dz = 8\int_0^1 \left((1-z)y - \frac{y^2}{2}\right)(1-z)z\Big|_{y=0}^{y=1-z} dz$$

$$= 8\int_0^1 \frac{(1-z)^3 z}{2}\,dz \underset{\substack{\uparrow \\ u=1-z}}{=} 8\int_1^0 \frac{u^3(1-u)}{2}\cdot(-1)\,du = 8\left(-\frac{u^4}{8} + \frac{u^5}{10}\right)\Big|_{u=1}^{u=0} = \boxed{\frac{1}{5}}$$

5. Each part is about partial derivatives. * The solution in the archive has an error.

(a) (2 points) Suppose $z = x^3y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial z}{\partial r}$.

$$\frac{\partial z}{\partial r} \underset{\uparrow}{=} \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \boxed{3x^2y^2\cdot\cos\theta + 2x^3y\sin\theta}$$

chain rule

(b) (3 points) Suppose $z = x^2 + xy + y^2$ and $x = g(u, v)$ and $y = h(u, v)$. Find $\frac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \boxed{(2x+y)\,g_u + (x+2y)\,h_u}$$

(c) (5 points) Suppose the equations

$$\begin{aligned} uv &= x^2 + y^2 \\ u^2 - v^2 &= xy \end{aligned}$$

implicitly define u, v as functions of x, y. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$. (Your answer can be in terms of u, v, x, y.)

$$uv = x^2+y^2 \Rightarrow \frac{\partial}{\partial x}(uv) = \frac{\partial}{\partial x}(x^2+y^2) \Rightarrow u_x v + u v_x = 2x \quad (*)$$

$$u^2-v^2 = xy \Rightarrow \frac{\partial}{\partial x}(u^2-v^2) = \frac{\partial}{\partial x}(xy) \Rightarrow 2uu_x - 2vv_x = y \quad (**)$$

We solve $(*)$ and $(**)$ for u_x and v_x.

$(*) \times 2v$: $2v^2u_x + 2uvv_x = 4vx$

$(**) \times u$: $2u^2u_x - 2uvv_x = uy$.

$$\Rightarrow 2(u^2+v^2)u_x = 4vx + uy \Rightarrow u_x = \frac{4vx+uy}{2(u^2+v^2)}$$

$(*) \times 2u$: $2uvu_x + 2u^2v_x = 4ux$

$(**) \times v$: $2uvu_x - 2v^2v_x = vy$

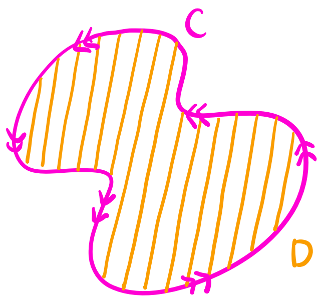
$$\Rightarrow 2(u^2+v^2)v_x = 4ux - vy \Rightarrow v_x = \frac{4ux-vy}{2(u^2+v^2)}$$

$$\Rightarrow \boxed{u_x = \frac{4vx+uy}{2(u^2+v^2)} \quad \text{and} \quad v_x = \frac{4ux-vy}{2(u^2+v^2)}}$$

6. Evaluate the line integral

$$\int_C -y\,dx + x\,dy$$

for the following closed curves C. In each case, the orientation of C is assumed to be counter-clockwise.



D : the region bounded by C

$\Rightarrow \partial D = C$ is positively oriented.

$P = -y,\ Q = x \Rightarrow \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1-(-1) = 2.$

$$\int_C -y\,dx + x\,dy = \int_{\partial D} P\,dx + Q\,dy = \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=2}\,dA = 2\,\mathrm{Area}(D).$$

(a) (3 points) C is the circle $x^2 + y^2 = 1$.

D is a disk of radius $1 \Rightarrow 2\,\mathrm{Area}(D) = 2\cdot\pi\cdot 1^2 = \boxed{2\pi}$

(b) (3 points) C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

D is given by $\dfrac{x^2}{4} + \dfrac{y^2}{9} \le 1 \Rightarrow 2\,\mathrm{Area}(D) = 2\cdot\pi\cdot 2\cdot 3 = \boxed{12\pi}$

<u>Note</u> As a general fact, the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$ is given by $ab\pi$. You can use this fact without proof on the exam. If you are still curious about the proof, you can see Fact 1 in the Final exam facts note.

(c) (4 points) C is the triangle with vertices $(-1, 1)$, $(1, -1)$, and $(-2, -2)$.

Set $P_1 = (-1,1,0)$, $P_2 = (1,-1,0)$, $P_3 = (-2,-2,0)$.

D is the triangular region with vertices at P_1, P_2, P_3

$\Rightarrow 2\,\mathrm{Area}(D) = 2\cdot\frac{1}{2}\left|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\right| = \left|(2,-2,0)\times(-1,-3,0)\right| = \boxed{8}$

7. Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = \left(x - \frac{x^3}{3}\right)\mathbf{i} + \left(y - \frac{y^3}{3}\right)\mathbf{j} + \left(z - \frac{z^3}{3}\right)\mathbf{k}.$$

(a) (1 point) Find the divergence $\nabla \cdot \mathbf{F}$.

$$P = x - \frac{x^3}{3}, \quad Q = y - \frac{y^3}{3}, \quad R = z - \frac{z^3}{3}.$$

$$\Rightarrow \operatorname{div}(\vec{F}) = P_x + Q_y + R_z = (1-x^2) + (1-y^2) + (1-z^2) = \boxed{3 - x^2 - y^2 - z^2}$$

(b) (4 points) If S is the surface $x^2 + y^2 + z^2 = 1$, find the flux

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

out of the surface S.

Let E be the ball $x^2+y^2+z^2 \le 1 \Rightarrow \partial E = S$ is oriented outward.

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial E} \vec{F} \cdot d\vec{S} \underset{\substack{\uparrow \\ \text{div. thm}}}{=} \iiint_E \operatorname{div}(\vec{F})\, dV \underset{\substack{\uparrow \\ \text{(a)}}}{=} \iiint_E 3 - x^2 - y^2 - z^2 \, dV$$

In spherical coordinates : $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, $0 \le \rho \le 1$ for E.

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\pi} \int_0^1 (3-\rho^2) \overset{\text{Jacobian}}{\underline{\rho^2 \sin\varphi}} \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left(\rho^3 - \frac{\rho^5}{5}\right) \sin\varphi \Big|_{\rho=0}^{\rho=1} d\varphi\, d\theta = \int_0^{2\pi} \int_0^{\pi} \frac{4}{5} \sin\varphi \, d\varphi\, d\theta$$

$$= \int_0^{2\pi} -\frac{4}{5}\cos\varphi \Big|_{\varphi=0}^{\varphi=\pi} d\theta = \int_0^{2\pi} \frac{8}{5}\, d\theta = \boxed{\frac{16\pi}{5}}$$

(c) (5 points) Find the equation of a closed surface S such that the flux out of the surface is maximum among all possible closed surfaces.

Let E be the solid bounded by $S \Rightarrow \partial E = S$ is oriented outward.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial E} \vec{F} \cdot d\vec{S} \underset{\substack{\uparrow \\ \text{div. thm}}}{=} \iiint_E \operatorname{div}(\vec{F})\, dV \underset{\substack{\uparrow \\ \text{(a)}}}{=} \iiint_E 3 - x^2 - y^2 - z^2 \, dV$$

The integral is maximized when E contains all points at which the integrand $3 - x^2 - y^2 - z^2$ is nonnegative.

$\Rightarrow E$ is given by $x^2+y^2+z^2 \le 3$.

$\Rightarrow S = \partial E$ is the sphere $\boxed{x^2+y^2+z^2 = 3}$

8. Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = 3z\mathbf{i} + (x + z^2/2)\mathbf{j} + (2y + yz)\mathbf{k}.$$

(a) (2 points) Evaluate the curl $\nabla \times \mathbf{F}$.

$P = 3z,\ Q = x + \frac{z^2}{2},\ R = 2y + yz$

$\Rightarrow \text{curl}(\vec{F}) = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y) = (2 + z - z,\ 3 - 0,\ 1 - 0) = \boxed{(2,3,1)}$

(b) (8 points) Evaluate the line integral (circulation)

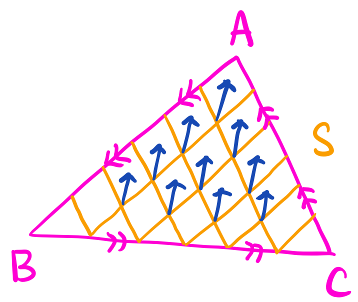
$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

with $\mathbf{r}$ being the position vector of a point on the closed curve C (as usual) and C is the triangle joining the points

$$A = (2, 1, -1),\ B = (1, 2, -2),\ C = (4, 0, -2).$$

The orientation of C is from A to B, B to ~~A~~ C, and then C back to A.

* The problem contains a typo as indicated above. In addition, the notations are confusing as C denotes a point and a curve.



S: the triangular surface bounded by C.

For positive orientation of $\partial S = C$, a normal vector of S is given by

$\overrightarrow{AB} \times \overrightarrow{AC} = (-1, 1, -1) \times (2, -1, -1) = (-2, -3, -1)$.

$\Rightarrow$ The unit normal vector of S is

$$\vec{n} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} = \frac{(-2,-3,-1)}{\sqrt{(-2)^2 + (-3)^2 + (-1)^2}} = \frac{1}{\sqrt{14}}(-2,-3,-1).$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} \underset{\substack{\uparrow \\ \text{Stokes' thm}}}{=} \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \iint_S \text{curl}(\vec{F}) \cdot \vec{n}\, dS.$$

$$\text{curl}(\vec{F}) \cdot \vec{n} = (2,3,1) \cdot \frac{1}{\sqrt{14}}(-2,-3,-1) = -\sqrt{14}.$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \iint_S -\sqrt{14}\, dS = \sqrt{14}\ \text{Area}(S) = -\sqrt{14} \cdot \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \boxed{-7}$$

9. Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + 0\mathbf{k}.$$

Evaluate the line integral (circulation)

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

for the following closed curves C:

(a) (2 points) C is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ with $0 \le t \le 2\pi$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\vec{F}(\vec{r}(t)) = (-\sin(t), \cos(t), 0), \quad \vec{r}\,'(t) = (-\sin(t), \cos(t), 0)$$

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \sin^2(t) + \cos^2(t) = 1$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$$

(b) (3 points) C is $\mathbf{r}(t) = \cos(2t)\,\mathbf{i} + \sin(2t)\,\mathbf{j} + \sin\left(\frac{t}{2}\right)\mathbf{k}$ with $0 \le t \le 2\pi$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\vec{F}(\vec{r}(t)) = (-\sin(2t), \cos(2t), 0), \quad \vec{r}\,'(t) = \left(-2\sin(2t), 2\cos(2t), \frac{1}{2}\cos\left(\frac{t}{2}\right)\right)$$

$$\Rightarrow \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 2\sin^2(2t) + 2\cos^2(2t) = 2$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 2\,dt = \boxed{4\pi}$$

(c) (5 points) C is $\mathbf{r}(t) = \cos\sqrt{t}\,\mathbf{i} + \sin\sqrt{t}\,\mathbf{j} + \sin\left(\frac{t}{2\pi}\right)\mathbf{k}$ with $0 \le t \le 64\pi^2$.

Set $u = \sqrt{t} \Rightarrow \vec{r}(u) = \left(\cos(u), \sin(u), \sin\left(\frac{u^2}{2\pi}\right)\right)$ with $0 \le u \le 8\pi$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{8\pi} \vec{F}(\vec{r}(u)) \cdot \vec{r}\,'(u)\,du$$

$$\vec{F}(\vec{r}(u)) = (-\sin(u), \cos(u), 0), \quad \vec{r}\,'(t) = \left(-\sin(u), \cos(u), \frac{u}{\pi}\cos\left(\frac{u^2}{2\pi}\right)\right)$$

$$\Rightarrow \vec{F}(\vec{r}(u)) \cdot \vec{r}\,'(u) = \sin^2(u) + \cos^2(u) = 1$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^{8\pi} 1\,dt = \boxed{8\pi}$$